

Generalized Snell Envelope as a Minimal Solution of BSDE With Lower Barriers

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Abstract

The aim of this paper is to characterize the snell envelope of a given \mathcal{P} -measurable process $l := (l_t)_{0 \leq t \leq T}$ as the minimal solution of some backward stochastic differential equation with lower general reflecting barriers and to prove that this minimal solution exists.

Keys Words: Backward stochastic differential equation; comparison theorem; Snell envelope.

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1 Introduction and notations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, P)$ be a stochastic basis on which is defined a Brownian motion $(B_t)_{t \leq T}$ such that $(\mathcal{F}_t)_{t \leq T}$ is the natural filtration of $(B_t)_{t \leq T}$ and \mathcal{F}_0 contains all P -null sets of \mathcal{F} . Note that $(\mathcal{F}_t)_{t \leq T}$ satisfies the usual conditions, *i.e.* it is right continuous and complete.

Let us first introduce the following notations :

- \mathcal{P} is the sigma algebra of \mathcal{F}_t -predictable sets on $\Omega \times [0, T]$.
- \mathcal{D} is the set of \mathcal{P} -measurable and right continuous with left limits (*rcll* for short) processes $(Y_t)_{t \leq T}$ with values in \mathbb{R} .

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• For a given process $Y \in \mathcal{D}$, we denote : $Y_{t-} = \lim_{s \nearrow t} Y_s, t \leq T$ ($Y_{0-} = Y_0$), and $\Delta_s Y = Y_s - Y_{s-}$ the size of its jump at time s .

- $\mathcal{K} := \{K \in \mathcal{D} : K \text{ is nondecreasing and } K_0 = 0\}$.
- $\mathcal{L}^{2,d}$ the set of \mathbb{R}^d -valued and \mathcal{P} -measurable processes $(Z_t)_{t \leq T}$ such that

$$\int_0^T |Z_s|^2 ds < \infty, P - a.s.$$

The aim of this paper is to characterize the snell envelope of a given \mathcal{P} -measurable process $l := (l_t)_{0 \leq t \leq T}$ as the minimal solution of some reflected BSDE with lower barriers (RBSDE for short).

Let $l := (l_t)_{0 \leq t \leq T}$ be an \mathcal{F}_t -adapted right continuous with left limits (*rcll* for short) process with values in \mathbb{R} of class $D[0, T]$, that is the family $(l_\nu)_{\nu \in \mathcal{T}}$ is uniformly integrable, where \mathcal{T} is the set of all \mathcal{F}_t -stopping times ν , such that $0 \leq \nu \leq T$. The Snell envelope $\mathcal{S}_t(l)$ of $l := (l_t)_{0 \leq t \leq T}$ is defined as

$$\mathcal{S}_t(l) = \text{ess sup}_{\nu \in \mathcal{T}_t} \mathbb{E}[l_\nu | \mathcal{F}_t], \quad (1.1)$$

where \mathcal{T}_t is the set of all stopping times valued between t and T . According to the work of Mertens (see [4]), \mathcal{S} is the smallest *rcll*-supermartingale of class $D[0, T]$ which dominates the process l , i.e., P -a.s., $\forall t \leq T, l_t \leq \mathcal{S}_t(l)$.

Suppose now that l is neither of class $D[0, T]$ nor a *rcll* process but just \mathcal{P} -measurable, it is natural to ask whether we can define the smallest local supermartingale which dominates the process l ? In order to give a positive answer to this question, let $L \in \mathcal{D}$ and $\delta \in \mathcal{K}$ and assume that there exists a local martingale $M_t = M_0 + \int_0^t \kappa_s dB_s$ such that P -a.s.,

$$L_t \leq M_t \text{ on } [0, T[\text{ and } l_t \leq M_t d\delta_t - a.e. \text{ on } [0, T] \text{ and } l_T \leq M_T.$$

Theorem 3.1 states that Y the minimal solution of the following RBSDE with lower barriers L and l ,

$$\left\{ \begin{array}{ll} (i) & Y_t = L_T + \int_t^T dK_s^+ - \int_t^T Z_s dB_s, t \leq T, \\ (ii) & \forall t \in [0, T[, L_t \leq Y_t, \\ (iii) & \text{on }]0, T[, l_t \leq Y_{t-}, d\delta_t - a.e. \\ (iv) & \forall L^* \in \mathcal{D} \text{ satisfying } \forall t < T, L_t \leq L_t^* \leq Y_t \text{ and} \\ & \text{on }]0, T[, l_t \leq L_{t-}^*, d\delta_t - a.e. \\ & \text{we have } \int_0^T (Y_{t-} - L_{t-}^*) dK_t^+ = 0, \text{ a.s.,} \\ (v) & Y \in \mathcal{D}, K^+ \in \mathcal{K}, Z \in \mathcal{L}^{2,d}, \end{array} \right. \quad (1.2)$$

is the smallest *rcll* local supermartingale satisfying

$$\forall t \in [0, T[, L_t \leq Y_t, l_t \leq Y_{t-} d\delta - a.e., \text{ on } [0, T] \text{ and } l_T \leq Y_T.$$

The process Y will be called later the generalized Snell envelope associated to L, l and δ and it will be denoted by $\mathcal{S}(L, l, \delta, l_T)$. It is worth mentioning here that when the process l is bounded and progressively measurable and δ is the Lebesgue measure, L. Stettner and J. Zabczyk characterize the strong Snell envelope V , which is the smallest right continuous non-negative supermartingale such that $V \geq l$, $dtdP$ -a.s., as the limit of some non-linear equation.

As by product, if we suppose that there exist $L \in \mathcal{D}$ and M a local martingale such that $L_t \leq l_t \leq M_t$, dt -a.e. and $l_T \leq M_T$. We prove that Y the minimal solution of the following reflected BSDE

$$\left\{ \begin{array}{ll} (i) & Y_t = L_T + \int_t^T dK_s^+ - \int_t^T Z_s dB_s, t \leq T, \\ (ii) & \text{on }]0, T], L_t \leq Y_t, dt - a.e \\ (iii) & \forall L^* \in \mathcal{D} \text{ satisfying } L_t \leq L_t^* \leq Y_t \text{ } dt - a.e. \text{ we have} \\ & \int_0^T (Y_{t-} - L_{t-}^*) dK_t^+ = 0, \text{ a.s.}, \\ (v) & Y \in \mathcal{D}, \quad K^+ \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \end{array} \right. \quad (1.3)$$

is the smallest *rcll* local supermartingale bounding the given process $l := (l_t)_{0 \leq t \leq T}$, *i.e.*

$$l_t \leq Y_t, dt - a.e \text{ and } l_T \leq Y_T.$$

We shall prove later that equation (1.2) has a minimal solution. We shall also characterize the solution Y as the generalized snell envelope $\mathcal{S}(L) = \mathcal{S}(L, l, \delta, L_T)$ and we shall show that the generalized snell envelope $\mathcal{S}(L, 0, 0, L_T)$ coincides with the usual snell envelope defined by equality (1.1) if the process L is of class $D[0, T]$.

We need also the following notations :

- For a set B , we denote by B^c the complement of B and 1_B denotes the indicator of B .
- For each $(a, b) \in \mathbb{R}^2$, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.
- For all $(a, b, c) \in \mathbb{R}^3$ such that $a \leq c$, $a \vee b \wedge c = \min(\max(a, b), c) = \max(a, \min(c, b))$.

Throughout the paper we introduce the following data :

- ξ is an \mathcal{F}_T -measurable one dimensional random variable.
- $L := \{L_t, 0 \leq t \leq T\}$ is a barrier which belongs to \mathcal{D} .
- $l := \{l_t, 0 \leq t \leq T\}$ is a \mathcal{P} -measurable process.
- $\delta \in \mathcal{K}$.
- $\mathcal{M} = \mathcal{M}(L, l, \delta, \xi)$ is the set of *rcll* local supermartingale $V_t = V_0 - A_t + \int_0^t \chi_s dB_s$, where $A \in \mathcal{K}$ and $\chi \in \mathcal{L}^{2,d}$ such that

$$L_t \leq V_t, l_t \leq V_{t-} d\delta_t - a.e. \text{ and } \xi \leq V_T.$$

We should note here that if $V_t = V_0 - A_t + \int_0^t \chi_s dB_s \in \mathcal{M}$, then we have

1. $V_t + 1 \in \mathcal{M}$.
2. $V_t + A_t = V_0 + \int_0^t \chi_s dB_s \in \mathcal{M}$.

2 Preliminaries

In view of clarifying this issue, we recall some results concerning generalized reflected BSDEs (GRBSDE for short) with two *rcll* obstacles. We present both the existence and comparison theorem for minimal

solutions of this kind of equations. Those results will play a crucial role in our proofs (see [2] for more details). We should note here that the notion of reflected BSDE with two obstacles has been first introduced by Civitanic and Karatzas [1].

2.1 Existence of a minimal solutions for GRBSDE

Let us recall first the following definition of two singular measures.

Definition 2.1. Let K^1 and K^2 be two processes in \mathcal{K} . We say that : K^1 and K^2 are singular if and only if there exists a set $D \in \mathcal{P}$ such that

$$\mathbb{E} \int_0^T 1_D(s, \omega) dK_s^1(\omega) = \mathbb{E} \int_0^T 1_{D^c}(s, \omega) dK_s^2(\omega) = 0.$$

This is denoted by $dK^1 \perp dK^2$.

Let us now define the notion of solution of the GRBSDE with two obstacles L and U . For this reason, let :

- $g : [0, T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that

$$\forall y \in \mathbb{R}, (t, \omega) \longmapsto g(t, \omega, L_{t-}(\omega) \vee y \wedge U_{t-}(\omega)) \text{ is } \mathcal{P} - \text{measurable.}$$

- $U := \{U_t, 0 \leq t \leq T\}$ be a barrier such that $L_t \leq U_t, \forall t \in [0, T]$.

Definition 2.2. 1. We say that $(Y, Z, K^+, K^-) := (Y_t, Z_t, K_t^+, K_t^-)_{t \leq T}$ is a solution of the generalized reflected BSDE, associated with the data (ξ, g, δ, L, U) , if the following hold :

$$\left\{ \begin{array}{ll} (i) & Y_t = \xi + \int_t^T g(s, Y_{s-}) d\delta_s + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, t \leq T, \\ (ii) & \forall t \in [0, T], L_t \leq Y_t \leq U_t, \\ (iii) & \int_0^T (Y_{t-} - L_{t-}) dK_t^+ = \int_0^T (U_{t-} - Y_{t-}) dK_t^- = 0, \text{ a.s.}, \\ (iv) & Y \in \mathcal{D}, K^+, K^- \in \mathcal{K}, Z \in \mathcal{L}^{2,d}, \\ (v) & dK^+ \perp dK^-. \end{array} \right. \quad (2.4)$$

2. We say that the GRBSDE (2.4) has a minimal solution $(Y_t, Z_t, K_t^+, K_t^-)_{t \leq T}$ if for any other solution $(Y'_t, Z'_t, K'^+_t, K'^-_t)_{t \leq T}$ of (2.4) we have for all $t \leq T, Y_t \leq Y'_t, P$ -a.s.

We introduce also the following assumption :

(H) The function g and the barrier U satisfy the following :

- (a) There exists $\beta \in L^0(\Omega, L^1([0, T], \delta(dt), \mathbb{R}_+))$ such that : $\forall y \in \mathbb{R}, |g(t, \omega, L_{t-}(\omega) \vee y \wedge U_{t-}(\omega))| \leq \beta_t(\omega), \delta(dt)P(d\omega)$ -a.e.
- (b) $\delta(dt)P(d\omega)$ -a.e., the function $y \longmapsto g(t, \omega, L_{t-}(\omega) \vee y \wedge U_{t-}(\omega))$ is continuous.
- (c) The barrier U is a *rcll* local supermartingale, i.e. there exist $\alpha \in \mathcal{K}$ and $\gamma \in \mathcal{L}^{2,d}$ such that $U_t = U_0 - \alpha_t + \int_0^t \gamma_s dB_s$.

The following theorem has already been proved in [2]. We should note here that the barriers L and U are *rcll*, the continuous case has been studied in [3].

Theorem 2.1. If assumption (H) holds then the GRBSDE (2.4) has a minimal solution.

2.2 Comparison theorem for minimal solutions

Let us now recall the following comparison theorem which plays a crucial rule in the proof of the existence of solutions for RBSDE. The proof of this comparison theorem is based on an exponential change and an approximation scheme, see [2]. Let (Y, Z, K^+, K^-) be the minimal solution for the following GRBSDE

$$\left\{ \begin{array}{ll} (i) & Y_t = \xi + \int_t^T g(s, Y_{s-}) d\delta_s + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, t \leq T, \\ (ii) & \forall t \in [0, T[, L_t \leq Y_t \leq U_t, \\ (iii) & \int_0^T (Y_{t-} - L_{t-}) dK_t^+ = \int_0^T (U_{t-} - Y_{t-}) dK_t^- = 0, \text{ a.s.}, \\ (iv) & Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\ (v) & dK^+ \perp dK^-. \end{array} \right. \quad (2.5)$$

Let (Y', Z', K'^+, K'^-) be a solution for the following GRBSDE

$$\left\{ \begin{array}{ll} (i) & Y'_t = \xi' + \int_t^T dA'_s + \int_t^T dK'_s{}^+ - \int_t^T dK'_s{}^- - \int_t^T Z'_s dB_s, t \leq T, \\ (ii) & \forall t \in [0, T[, L'_t \leq Y'_t \leq U'_t, \\ (iii) & \int_0^T (Y'_{t-} - L'_{t-}) dK'_t{}^+ = \int_0^T (U'_{t-} - Y'_{t-}) dK'_t{}^- = 0, \text{ a.s.}, \\ (iv) & Y' \in \mathcal{D}, \quad K'^+, K'^- \in \mathcal{K}, \quad Z' \in \mathcal{L}^{2,d}, \\ (v) & dK'^+ \perp dK'^-, \end{array} \right. \quad (2.6)$$

where A' is a process in \mathcal{K} , L' and U' are two barriers which belong to \mathcal{D} such that $L'_t \leq U'_t, \forall t \in [0, T[$. Assume moreover that for every $t \in [0, T]$

- (a) $\xi \leq \xi'$.
- (b) $Y'_t \leq U_t, L'_t \leq Y_t, \forall t \in [0, T[$.
- (c) $g(s, Y'_{s-}) d\delta_s \leq dA'_s$ on $[0, T]$.

Theorem 2.2. (*Comparison theorem for minimal solutions, see [2]*) Assume that the above assumptions hold then we have :

1. $Y_t \leq Y'_t$, for every $t \in [0, T]$, P -a.s.
2. $1_{\{U'_{t-}=U_{t-}\}} dK_t^- \leq dK_t'^-$ and $1_{\{L'_{t-}=L_{t-}\}} dK_t'^+ \leq dK_t^+$.

3 Generalized Snell envelope as a solution of some RBSDE

In this section, we prove an existence result of a minimal solution for some reflected BSDE with lower barriers. We shall also characterize this minimal solution Y as the smallest *rcll* local supermartingale satisfying

$$\forall t \in [0, T[, L_t \leq Y_t, l_t \leq Y_{t-} d\delta_t - a.e., \text{ on } [0, T] \text{ and } \xi \leq Y_T.$$

Let us now introduce the definition of our RBSDE with lower obstacles.

Definition 3.1. 1. We call $(Y, Z, K^+) := (Y_t, Z_t, K_t^+)_{t \leq T}$ a solution of the RBSDE, associated with the data (ξ, L, l, δ) , if the following hold :

$$\left\{ \begin{array}{ll} (i) & Y_t = \xi + \int_t^T dK_s^+ - \int_t^T Z_s dB_s, t \leq T, \\ (ii) & \forall t \in [0, T[, L_t \leq Y_t, \\ (iii) & \text{on }]0, T], l_t \leq Y_{t-}, d\delta_t - a.e. \\ (iv) & \forall L^* \in \mathcal{D} \text{ satisfying } \forall t < T, L_t \leq L_t^* \leq Y_t \text{ and} \\ & \text{on }]0, T], l_t \leq L_{t-}^*, d\delta_t - a.e. \\ & \text{we have } \int_0^T (Y_{t-} - L_{t-}^*) dK_t^+ = 0, a.s., \\ (v) & Y \in \mathcal{D}, K^+ \in \mathcal{K}, Z \in \mathcal{L}^{2,d}. \end{array} \right. \quad (3.7)$$

2. We say that the RBSDE (3.7) has a minimal solution $(Y_t, Z_t, K_t^+)_{t \leq T}$ if for any other solution $(Y'_t, Z'_t, K'^+_{t \leq T})$ of (3.7) we have for all $t \leq T$, $Y_t \leq Y'_t$, P -a.s.

3.1 Main result

Let $L \in \mathcal{D}$, $\xi \in L^0(\Omega)$, $l \in L^0(\Omega \times [0, T])$ and $\delta \in \mathcal{K}$. We assume the following hypothesis :

(A) There exists a local martingale $M_t = M_0 + \int_0^t \kappa_s dB_s$ such that P -a.s., $L_t \leq M_t$ on $[0, T[$ and $l_t \leq M_t d\delta_t - a.e.$ on $[0, T]$ and $\xi \leq M_T$. This is equivalent to $\mathcal{M} \neq \emptyset$.

The main result of this paper is the following.

Theorem 3.1. If assumption (A) hold then the RBSDE (3.7) has a minimal solution $(Y_t, Z_t, K_t^+)_{t \leq T}$. Moreover Y is the smallest rcll local supermartingale satisfying

$$\forall t \in [0, T[, L_t \leq Y_t, l_t \leq Y_{t-} d\delta_t - a.e., \text{ on } [0, T] \text{ and } \xi \leq Y_T.$$

We say that Y is the generalized Snell envelope associated to L, l, δ and ξ . We denote it by $\mathcal{S}(L, l, \delta, \xi)$.

3.1.1 Auxiliary penalized equation

Let $V_t = V_0 - A_t + \int_0^t \chi_s dB_s \in \mathcal{M}$. Let also $(Y^{(n,V)}, Z^{(n,V)}, K^{(n,V)+}, K^{(n,V)-})$ be the minimal solution of the following penalized RBSDE with two rcll barriers

$$\left\{ \begin{array}{ll} (i) & Y_t^{(n,V)} = \xi + n \int_t^T (l_s - Y_{s-}^{(n,V)})^+ d\delta_s + \int_t^T dK_s^{(n,V)+} \\ & - \int_t^T dK_s^{(n,V)-} - \int_t^T Z_s^{(n,V)} dB_s, t \leq T, \\ (ii) & \forall t \in [0, T[, L_t \leq Y_t^{(n,V)} \leq V_t, \\ (iii) & \int_0^T (Y_{t-}^{(n,V)} - L_{t-}) dK_t^{(n,V)+} = \int_0^T (V_{t-} - Y_{t-}^{(n,V)}) dK_t^{(n,V)-} = 0, P - a.s., \\ (iv) & Y^{(n,V)} \in \mathcal{D}, K^{(n,V)+}, K^{(n,V)-} \in \mathcal{K}, Z^{(n,V)} \in \mathcal{L}^{2,d}, \\ (v) & dK^{(n,V)+} \perp dK^{(n,V)-}. \end{array} \right. \quad (3.8)$$

We should mention here that the minimal solution to (3.8) exists according to Theorem 3.1 (see [2] for the proof).

Our objective now is to prove that $Y^{(n,V)}$ does not depend on $V \in \mathcal{M}$ and converges to some Y which belongs to \mathcal{M} . This means that the process Y is the smallest *rcll* local supermartingale satisfying

$$\forall t \in [0, T[, L_t \leq Y_t, l_t \leq Y_{t-} \quad d\delta_t - a.e., \text{ on } [0, T] \text{ and } \xi \leq Y_T.$$

It follows from comparison theorem 2.2, applied to $Y^{(n,V)}$ and V_t (we can also apply Tanaka's formula to the process $(V_t - Y_t^{(n,V)})^+ = (V_t - Y_t^{(n,V)})$), that for every $n \in \mathbb{N}$ $dK^{(n,V)-} = 0$. Hence $(Y^{(n,V)}, Z^{(n,V)}, K^{(n,V)+})$ is the minimal solution of the following GRBSDE

$$\left\{ \begin{array}{l} (i) \quad Y_t^{(n,V)} = \xi + n \int_t^T (l_s - Y_{s-}^{(n,V)})^+ d\delta_s + \int_t^T dK_s^{(n,V)+} \\ \quad - \int_t^T Z_s^{(n,V)} dB_s, t \leq T, \\ (ii) \quad \forall t \in [0, T[, L_t \leq Y_t^{(n,V)}, \\ (iii) \quad \int_0^T (Y_{t-}^{(n,V)} - L_{t-}) dK_t^{(n,V)+} = 0, P - a.s., \\ (iv) \quad Y^{(n,V)} \in \mathcal{D}, \quad K^{(n,V)+} \in \mathcal{K}, \quad Z^{(n,V)} \in \mathcal{L}^{2,d}. \end{array} \right. \quad (3.9)$$

Moreover, for every $V \in \mathcal{M}$ and all $(n, t) \in \mathbb{N} \times [0, T]$, $Y_t^{(n,V)} \leq V_t$.

Since $Y^{(n,M)}$ is also the minimal solution of (3.9), then for every V , $Y^{(n,V)} = Y^{(n,M)}$. From now on we denote the solution of (3.9) by (Y^n, Z^n, K^{n+}) .

Now by using comparison theorem 2.2 we get, for every $V \in \mathcal{M}$, that

$$L_t \leq Y_t^n \leq Y_t^{n+1} \leq V_t. \quad (3.10)$$

Now let us set

$$Y_t = \sup_n Y_t^n \text{ and } Y_t^- = \sup_n Y_{t-}^n. \quad (3.11)$$

The following results guarantee that the process Y is the smallest *rcll* local supermartingale satisfying

$$\forall t \in [0, T[, L_t \leq Y_t, l_t \leq Y_{t-} \quad d\delta_t - a.e., \text{ on } [0, T] \text{ and } \xi \leq Y_T.$$

By letting n to infinity in (3.10) and using assumption (A) we have the following.

Lemma 3.1. *For every $t \in [0, T]$ we have for every $V \in \mathcal{M}$,*

$$L_t \leq Y_t \leq V_t \quad \text{on } [0, T[\quad \text{and} \quad L_{t-} \leq Y_t^- \leq V_{t-} \quad \text{on }]0, T].$$

Proposition 3.1. *The process Y defined by (3.11) satisfy the following properties :*

1. *Y is a *rcll* local supermartingale and $Y_t^- \leq Y_{t-}$, for every $t \in]0, T]$.*
2. *$l_t \leq Y_t^-$, $d\delta_t - a.e.$, on $]0, T]$.*

In particular it follows that Y belongs to \mathcal{M} .

Proof. 1. Recall that $M_t = M_0 + \int_0^t \kappa_s dB_s \in \mathcal{M}$. We have

$$\begin{aligned} Y_t^n - M_t &= \xi - M_T + n \int_t^T (l_s - Y_{s-}^n)^+ d\delta_s + \int_t^T dK_s^{n+} + \int_t^T (\bar{Z}_s^n - \kappa_s) dB_s. \end{aligned}$$

Let $(\tau_i)_{i \geq 1}$ be the family of stopping times defined by

$$\tau_i = \inf\{s \geq 0 : M_s - L_s \geq i + M_0 - L_0\} \wedge T. \quad (3.12)$$

Note that $\tau_i > 0$, P -a.s., for every $i \geq 1$. By using a localization procedure we have for every $i \geq 1$ and $n \geq 0$

$$\mathbb{E}(M_0 - Y_0^n) + n\mathbb{E} \int_0^{\tau_i-} (l_s - Y_{s-}^n)^+ d\delta_s + \mathbb{E}K_{\tau_i-}^{n+} \leq i + \mathbb{E}(M_0 - L_0). \quad (3.13)$$

Put

$$\begin{aligned} \mathcal{M}_t^n &= Y_t^n - M_t, \\ {}^i\mathcal{M}_t^n &= \mathcal{M}_t^n 1_{\{t < \tau_i\}} + \mathcal{M}_{\tau_i-}^n 1_{\{t \geq \tau_i\}}, \end{aligned} \quad (3.14)$$

we have

$$-i - \mathbb{E}(M_0 - L_0) \leq {}^i\mathcal{M}_t^n \leq 0 \quad \text{and} \quad {}^i\mathcal{M}_t^n \leq {}^i\mathcal{M}_t^{n+1} \quad \text{and} \quad t \rightarrow {}^i\mathcal{M}_t^n \text{ is a } rcll \text{ supermartingale.}$$

It follows then from Dellacherie and Meyer [4] that $\sup_n {}^i\mathcal{M}_t^n$ is also a *rcll* process supermartingale process. Since $P\left[\bigcup_{i=1}^{\infty} (\tau_i = T)\right] = 1$, it follows that Y_t is a *rcll* local supermartingale on $[0, T]$.

Now since for every $s \in]0, T]$ and $n \in \mathbb{N}$, $Y_{s-}^n \leq Y_{s-}$, it follows that $Y_s^- \leq Y_{s-}$.

2. On another hand, by letting n to infinity in inequality (3.13) and using Fatou's lemma it follows that

$$\mathbb{E} \int_0^{\tau_i-} (l_s - Y_{s-}^-)^+ d\delta_s = 0.$$

Hence

$$(l_s - Y_{s-}^-)^+ = 0 \quad d\delta_s - a.e. \text{ on } [0, T[.$$

Assume now that $Y_T^- < l_T$ and $\Delta_T \delta > 0$. It follows from [2], that for every $V \in \mathcal{M}$

$$Y_{T-}^n = L_{T-} \vee [\xi + n(l_T - Y_{T-}^n)^+ \Delta_T \delta] \wedge V_{T-} \geq [\xi + n(l_T - Y_{T-})^+ \Delta_T \delta] \wedge V_{T-}.$$

We get $Y_T^- = V_T$, which is absurd since $V_t + 1 \in \mathcal{M}$. Consequently

$$l_s \leq Y_{s-}^- \quad d\delta_s - a.e. \text{ on } [0, T].$$

The proof of Proposition 3.1 is finished. ■

3.1.2 Proof of the main result

Proof of Theorem 3.1. Let $L^* \in \mathcal{D}$ be such that for every $t \in [0, T]$, $L_t \leq L_t^* \leq Y_t$ and $l_t \leq L_{t-}^* d\delta_t - a.e.$. Let also (Y^*, Z, K^+, K^-) , which exists according to Theorem 3.1, the minimal solution of the following RBSDE

$$\begin{cases} (i) & Y_t^* = \xi + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, t \leq T, \\ (ii) & \forall t \in [0, T[, L_t^* \leq Y_t^* \leq Y_t, \\ (iii) & \int_0^T (Y_{t-} - Y_{t-}^*) dK_t^- = \int_0^T (Y_{t-}^* - L_{t-}^*) dK_t^+ = 0, \text{ a.s.}, \\ (v) & Y^* \in \mathcal{D}, K^+, K^- \in \mathcal{K}, Z \in \mathcal{L}^{2,d}, \\ (vi) & dK^+ \perp dK^-. \end{cases} \quad (3.15)$$

By the same argument as before with $V = Y$ (Y is the process defined in the previous subsection), one can see that $dK^- = 0$, hence $Y^* \in \mathcal{M}$. By Lemma 3.1 and (ii) of Equation (3.15) we get

$$Y_s^* = Y_s.$$

Henceforth

$$(Y_{t-} - L_{t-}^*) dK_t^+ = 0.$$

Consequently, for every $V \in \mathcal{M}$, (Y, Z, K^+) is a solution of (3.7). Moreover the process Y is the smallest *rcll* local supermartingale satisfying

$$\forall t \in [0, T[, L_t \leq Y_t, l_t \leq Y_{t-} d\delta_t - a.e., \text{ on } [0, T] \text{ and } \xi \leq Y_T.$$

■

As by product we obtain the following theorem.

Theorem 3.2. Let $(T_i)_{i \geq 1}$ be a sequence of stopping times such that $[|T_i|] \cap [|T_j|] = \emptyset, \forall i \neq j$ and $\bigcup_{i \geq 1} [|T_i|] = \{(t, \omega) \in]0, T] \times \Omega : \Delta_t \delta(\omega) > 0\}$. Under assumption **(A)**, Y the minimal solution of (3.7) is the smallest *rcll* local supermartingale satisfying P -a.s.

$$\forall t \in [0, T[, L_t \leq Y_t, l_t \leq Y_{t-} d\delta_t^c - a.e., \text{ on } [0, T], \forall i \geq 1, l_{T_i} \leq Y_{T_i-} \text{ and } \xi \leq Y_T.$$

3.1.3 Some properties of the generalized Snell envelope

The generalized Snell envelope $Y = \mathcal{S}.(L, l, \delta, \xi)$ solution of RBSDE (3.7) has the following properties whose proofs are immediate.

Corollary 3.1. 1. $\mathcal{S}.(L, l, \delta, \xi) = \mathcal{S}.(L, \bar{l}, \delta, \xi)$, with $\bar{l}_s = l_s \vee L_{s-}$.

2. If $L' \leq L, d\delta' \ll d\delta, l' \leq l, d\delta' \text{ a.e.}, \xi' \leq \xi$ then (L', l', δ', ξ') satisfies condition **(A)** and $\mathcal{S}.(L', l', \delta', \xi') \leq \mathcal{S}.(L, l, \delta, \xi)$.

3. $\mathcal{S}.(L, l, \delta, \xi) \geq \mathcal{S}(L^\xi)$ (with equality if $l_t \leq L_{t-} d\delta_t - a.e., \text{ on } [0, T]$) where $\mathcal{S}(L^\xi) = \mathcal{S}.(L, 0, 0, \xi)$ and $L_t^\xi = L_t 1_{\{t < T\}} + \xi 1_{\{t = T\}}$.

4. Put $Y = \mathcal{S}.(L, l, \delta, \xi)$. If

$$l_t \leq l'_t \leq Y_{t-}, d\delta - a.e., \text{ on } [0, T] \text{ and } L_t \leq L'_t \leq Y_t, \forall t \in [0, T[, \text{ and } d\delta \sim d\delta',$$

then $\mathcal{S}(L, l, \delta, \xi) = \mathcal{S}(L', l', \delta', \xi)$.

In particular for every $L^* \in \mathcal{D}$ such that P -a.s.,

$$L_t \leq L_t^* \leq Y_t, \forall t \in [0, T[, \text{ and } l_t \leq L_{t-}^* \leq Y_{t-}, \text{ } d\delta_t - a.e., \text{ on } [0, T] \text{ and } L_T^* = \xi$$

we have $\mathcal{S}(L, l, \delta, \xi) = \mathcal{S}(L^*)$.

Remark 3.1. We know that if L is of class D then L satisfies assumption **(A)** (see Dellacherie-Meyer [4]). In this case our generalized snell envelope $\mathcal{S}(L) = \mathcal{S}(L, 0, 0, L_T)$ coincides with the usual snell envelope $\text{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}[L_\tau | \mathcal{F}_t]$, where \mathcal{T}_t is the set of all stopping times valued between t and T , as presented in Dellacherie-Meyer [4] and studied by several authors.

Example 3.1. If $\delta_t = t$ and there exist $L \in \mathcal{D}$ and M a local martingale such that $L_t \leq l_t \leq M_t$ and $\xi \leq M_T$. Let (Y, Z, K^+) be the minimal solution of the following RBSDE

$$\left\{ \begin{array}{ll} (i) & Y_t = \xi + \int_t^T dK_s^+ - \int_t^T Z_s dB_s, t \leq T, \\ (ii) & \text{on }]0, T], \text{ } l_t \leq Y_t, \text{ } dt - a.e \\ (iii) & \forall L^* \in \mathcal{D} \text{ satisfying } l_t \leq L_t^* \leq Y_t \text{ } dt - a.e. \text{ we have} \\ & \int_0^T (Y_{t-} - L_{t-}^*) dK_t^+ = 0, \text{ } a.s., \\ (v) & Y \in \mathcal{D}, \quad K^+ \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \end{array} \right.$$

Then Y is the smallest local supermartingale such that

$$l_t \leq Y_t, \text{ } dt - a.e \text{ and } \xi \leq Y_T.$$

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